$$=-\frac{t}{n}(\sin\tfrac{1}{2}\pi n+\sin\tfrac{3}{2}\pi n)=0.$$

$$\therefore f(x)=\tfrac{5}{2}t-(t/\pi)(4\sin x+2\sin 2x+\tfrac{4}{3}\sin 3x+\tfrac{4}{5}\sin 5x+\tfrac{4}{6}\sin 6x+\tfrac{4}{7}\sin 7x+\tfrac{4}{9}\sin 9x+\ldots)$$

$$\therefore\ V=e^{-hT}[\tfrac{5}{2}t-(t/\pi)(4\sin x e^{-kT}+2\sin 2x e^{-2^2kT}+\tfrac{4}{3}\sin 3x e^{-3^2kT}+\tfrac{4}{5}\sin 5x e^{-5^2kT}+\ldots)]$$

AVERAGE AND PROBABILITY.

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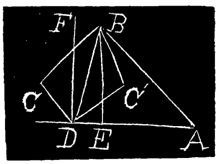
108. Proposed by A. H. HOLMES, Brunswick, Me.

Required the average area of the quadrilateral whose sides are a, b, c, and d.

I. Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $ABCD$ be the quadrilateral, $AB=a$, $AD=b$, $BC=c$, $CD=d$, $a>b>c>d$, $BE=v$, $DE=u$,.



When the quadrilateral is convex, the average area is, (the average area of ABD) + (the average area of BCD).

When the quadrilateral is concave, the average area is, (the average area of ABD) − (average area of $BC'D$).

Since BCD is equal to $BC'D$, the average area required is the average area of $ABD=\triangle$. Area $ABD=\frac{1}{2}ab\sin A$.

Now $DB=\sqrt{(u^2+v^2)}$.

$\therefore\ \cos A=(a^2+b^2-u^2-v^2)/2ab$.

$\therefore$ Area $ABD=\frac{1}{4}\sqrt{[4a^2b^2-(a^2+b^2-u^2-v^2)^2]}$.

But $v^2=a^2-(b-u)^2$.

$\therefore$ Area $ABD=\frac{1}{2}b\sqrt{[a^2-(b-u)^2]}=C$.

The limits of u are 0 and $[(c+d)^2+b^2-a^2]/2b=u'$.

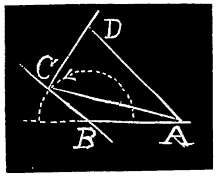
$$\therefore \triangle=\frac{\int_0^{u'}C\,du}{\int_0^{u'}du}=\frac{b}{2u'}\int_0^{u}\sqrt{[a^2-(b-u)^2]}du.$$

$$\therefore \triangle=\frac{b^2}{(c+d)^2+b^2-a^2}\left[\frac{b}{2}\sqrt{(a^2-b^2)}+\frac{a^2}{2}\sin^{-1}\frac{b}{a}-\frac{a^2+b^2-(c+d)^2}{8b^2}\right.$$

$$\left.\sqrt{\{4a^2b^2-[a^2+b^2-(c+d)^2]^2\}}-\frac{a^2}{2}\sin^{-1}\left(\frac{a^2+b^2-(c+d)^2}{2ab}\right)\right].$$

II. Solution by F. P. MATZ. Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance. O.

Let $ABCD$ be the quadrilateral, side $AB=a$, $BC=b$, $CD=c$, $DA=d$, $\angle ABC=\theta$, and $\angle CDA=\phi$.



Suppose the vertices of the quadrilateral to be movable, or hinged; then the side BC may make a complete revolution about B as a center so long as $(c+d)$ is not less than $(a+b)$. That is, for $(c+d)$ not less than $(a+b)$, the angle θ will vary uniformly from 0° to 360°; but as soon as the side BC has moved below the side AB produced, we have hour-glass quadrilaterals composed of two practically isolated triangles which are not to be considered in finding the average area of the quadrilateral (a, b, c, d). We are, therefore, constrained to regard θ as varying uniformly from 0° to 180°; but if $(c+d)<(a+b)$, which becomes certain when the sides of the quadrilateral (a, b, c, d) are numerically expressed, we are constrained to regard θ as varying uniformly from 0° to $\cos^{-1}\{[a^2+b^2-(c+d)^2]/2ab\}$.

From the diagram it is evident that $Q=(\triangle ABC+\triangle CDA)=\frac{1}{2}(ab\sin\theta+cd\sin\phi)$. Now $AC=\sqrt{(a^2+b^2-2ab\cos\theta)}$, and

$$\cos\phi=\frac{c^2+d^2-(a^2+b^2-2ab\cos\theta)}{2cd}.$$

[For the sake of brevity, put $m=c^2+d^2-a^2-b^2$, $n=4c^2d^2-m^2$, $p=n/4a^2b^2$, and $q=m/ab$].

$$\therefore\ \sin\phi=\sqrt{1-\left(\frac{m+2ab\cos\theta}{2cd}\right)^2}=\sqrt{\frac{(4c^2d^2-m^2)-4abm\cos\theta-4a^2b^2\cos^2\theta}{4c^2d^2}}$$

$$=\frac{1}{2cd}\sqrt{(n-4abm\cos\theta-4a^2b^2\cos^2\theta)}=(ab/cd)\sqrt{(p-q\cos\theta-\cos^2\theta)}$$

$$=(ab/cd)\sqrt{[(p+\tfrac{1}{4}q^2)-(\tfrac{1}{2}q+\cos\theta)^2]}.$$

Also, $Q=\frac{1}{2}ab\{\sin\theta+\sqrt{[(p+\frac{1}{4}q^2)-(\frac{1}{2}q+\cos\theta)^2]}\}$.

Representing $\cos^{-1}\{[a^2+b^2-(c+d)^2]/2ab\}$ by θ_1, the expression for the average area of the quadrilateral on the hypothesis that the interior angle at B vary uniformly from 0 to θ_1 becomes

$$Q_B=\frac{ab}{2\theta_1}\int_0^{\theta_1}\{\sin\theta+\sqrt{[(p+\tfrac{1}{4}q^2)-(\tfrac{1}{2}q+\cos\theta)^2]}\}d\theta.$$

Similar operations give Q_C, Q_D, and Q_A; and, therefore, the required average area of the quadrilateral (a, b, c, d) becomes $\mathbf{Q}=\frac{1}{4}(\mathbf{Q}_A+\mathbf{Q}_B+\mathbf{Q}_C+\mathbf{Q}_D)$. Slight modifications, in signs, etc., may be occasioned by quadrilaterals having a re-entrant angle.